Date: 16.1.2021

STABILITY

Concept of stability:

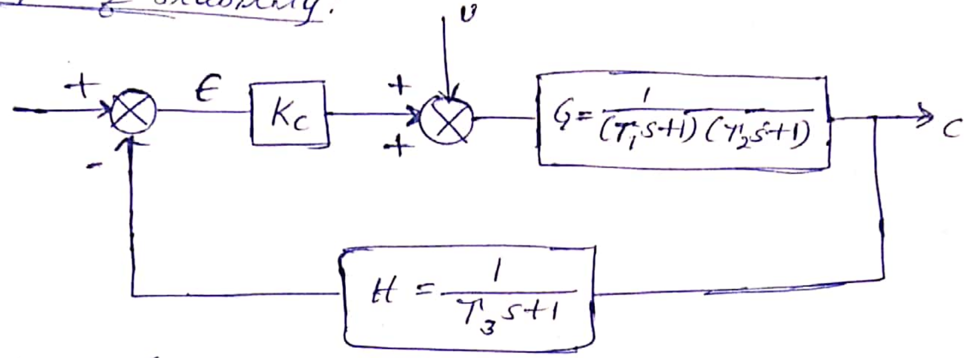


Fig- Third-order control system.

Consider the problem of stability in a control system slightly more complicated. This system represent proportional control of two stirred-tank heaters with measuring lag. Here only set-point changes are considered.

The overall transfer function from fig,

$$\frac{C}{R} = \frac{K_c G}{1 + K_c G H} \quad \text{——} (1)$$

In terms of the particular transfer functions,

$$\frac{C}{R} = \frac{K_c (T_3 s+1)}{(T_1 s+1)(T_2 s+1)(T_3 s+1) + K_c} \quad \text{——} (2)$$

Denominator of eq(2) is third-order.

For a unit-step change in R, the transform of the response is,

$$C = \frac{1}{s} \frac{K_c (T_3 s+1)}{(T_1 s+1)(T_2 s+1)(T_3 s+1) + K_c} \quad \text{——} (3)$$

To obtain the transient response C(t), it is necessary to find the inverse of eq.(3). which requires obtaining the roots of the denominator of eq.(2), which is third-order. Roots of denominator depend on the particular values of the time constants and K_c. These roots determine the nature of the transient response. The nature of the response for the control system is examined by varying the value of K_c and assuming the time constants τ_1, τ_2 and τ_3 to be fixed value.

To be specific, consider the step response for $\tau_1 = 1$, $\tau_2 = \frac{1}{2}$ and $\tau_3 = \frac{1}{3}$ for several values of K_c. The results of inversion of eq ③ are shown as response curves in fig.

Fig. 14-2
P.g. No. 165

From these response curves, it is observed that

As K_c increases, the system response becomes more oscillatory. Beyond a certain value of K_c, the successive amplitudes of the response grow rather than decay, this type of response is called "unstable".

As control system designers, clearly interested to determine quickly the values of K_c that give unstable responses, such as that corresponding to $K_c = 12$.

Date: 16.1.2021

Definition of Stability (Linear Systems)

Stable System - A system is defined as one for which the output response is bounded for all bounded inputs.

Unstable System - A system exhibiting an unbounded response to a bounded input is unstable.

A bounded input function is a function of time that always falls within certain bounds during the course of time.

Ex - The step function and sinusoidal function are bounded inputs. The function $f(t) = t$ is unbounded.

Although the definition of an unstable system states that the output becomes unbounded, this is true only in mathematical sense. An actual physical system always exhibits bounds or restraints. A linear mathematical model (set of linear differential equations describing the system) from which stability information is obtained is meaningful only over a certain range of variables.

Ex - A linear control valve gives a linear relation b/w flow and valve-top pressure only over the range of pressure (or flow) corresponding to values b/w which the valve is shut tight or wide open.

When the valve is wide open, for example, further change in pressure to the diaphragm will not increase the flow. Such a limitation is termed as saturation.

A physical system, when unstable, may not follow the response of its linear mathematical model beyond certain physical bounds but rather may saturate.

Scanned with CamScanner

Stability Criterion:

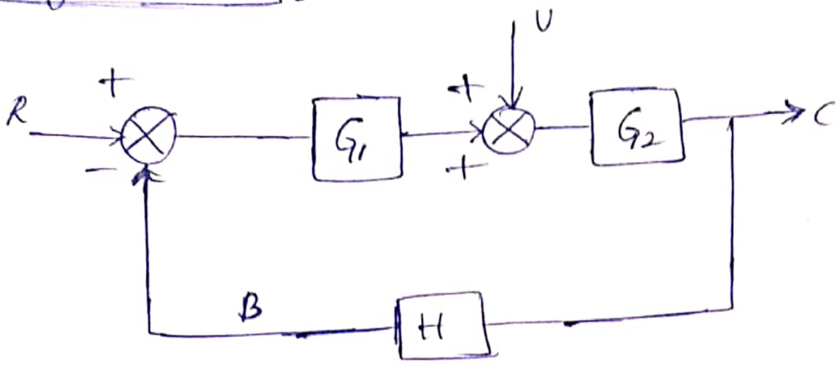


Fig - Basic Single-loop control system

Characteristic Equation:

From the above block-diagram of the control system

$$C = \frac{G_1 G_2}{1 + G_1 G_2 H} R + \frac{G_2}{1 + G_1 G_2 H} U \quad \text{——— (4)}$$

Let $G = G_1 G_2 H$

G - the open-loop transfer function because it relates the measured variable B to the set point 'R' if the feedback loop of fig. is disconnected from the comparator (i.e. if the loop is opened).

In terms of the open-loop transfer function G, eq (4) becomes,

$$\therefore \; C = \frac{G_1 G_2}{1+G} R + \frac{G_2}{1+G} U \quad \text{——— (5)}$$

If forcing functions $R(s)$ and $U(s)$ are given, then eq (5) is inverted to give the control system response.

To determine the conditions for which the system is stable, its necessary to test the response to a bounded input. Suppose a unit-step change in set point is applied. Then

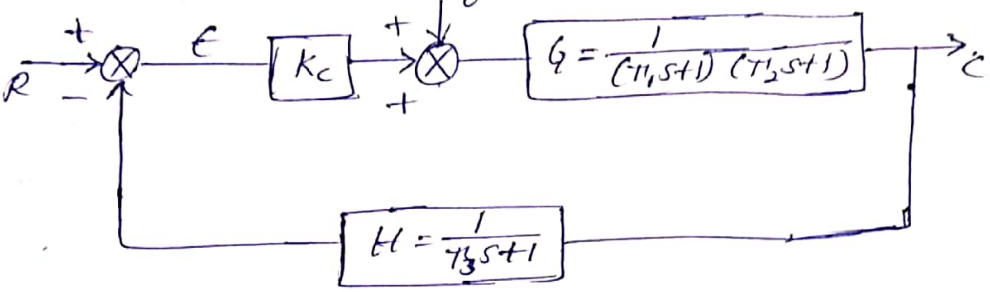
$$C(s) = \frac{G_1 G_2}{1+G} \frac{1}{s} = \frac{G_1 G_2 F(s)}{s(s-r_1)(s-r_2)\ldots\ldots(s-r_n)} \quad \text{—— (6)}$$

where, $r_1, r_2, \ldots\ldots r_n$ are the n roots of the equation.

$$1 + G(s) = 0 \qquad \text{—— (7)}$$

$F(s)$ – is a function that arises the rearrangement to the right-hand form of eq (6). Eq. (7) is called the "Characteristic Equation" for the control system of fig.

Ex – For the control system of fig, the step response is



$$C(s) = \frac{G_1 G_2}{s(1+G)}$$

$$= \frac{K_c}{(T_1 s+1)(T_2 s+1)} \Big/ s\left[1 + \frac{K_c}{(T_1 s+1)(T_2 s+1)(T_3 s+1)}\right]$$

which may be rearranged to

$$(8) \quad C(s) = \frac{K_c (T_3 s+1)}{s[(T_1 s+1)(T_2 s+1)(T_3 s+1) + K_c]}$$

$$= \frac{K_c (T_3 s+1)}{s[T_1 T_2 T_3 s^3 + (T_1 T_2 + T_1 T_3 + T_2 T_3)s^2 + (T_1 + T_2 + T_3)s + (1 + K_c)]}$$

This is equivalent to

$$C(s) = \frac{K_c (T_3 s+1) / T_1 T_2 T_3}{s(s-r_1)(s-r_2)(s-r_3)}$$

where, r_1, r_2 and r_3 are the roots of the characteristic eqn.

$$T_1 T_2 T_3 s^3 + (T_1 T_2 + T_1 T_3 + T_2 T_3)s^2 + (T_1 + T_2 + T_3)s + (1 + K_c) = 0 \quad \text{— (8)}$$

$F(s)$ in eq (6) is

$$F(s) = \frac{(T_1 s+1)(T_2 s+1)(T_3 s+1)}{T_1 T_2 T_3} \qquad \text{—— (*)}$$

* If there are any of the roots $r_1, r_2, \ldots\ldots r_n$ are in the right half of the complex plane, the response $C(t)$ will contain a term that grows exponentially in time and the system is unstable.

* If there are one or more roots of the characteristic eqn. at the origin, there is an s^m in the denominator of eq.⑥ (where $m \geq 2$) and the response is again unbounded, growing as a polynomial in time. This condition specifies m as greater than or equal to 2, but not 1, because one of the s terms in the denominator is accounted for by the fact that the input is a unit-step ($1/s$) in eq.⑥.

* If there is a pair of conjugate roots on the imaginary axis, the contribution to the overall step response is a pure sinusoid, which is bounded.

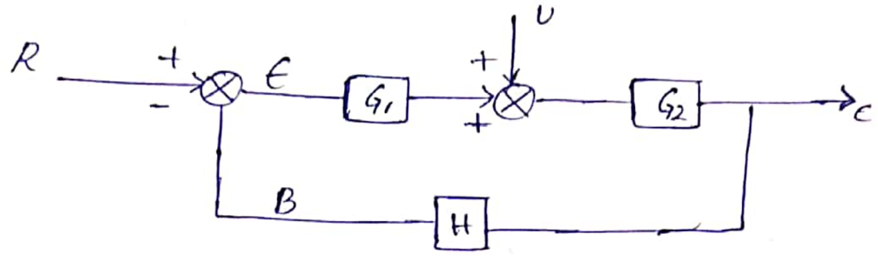
If the bounded input is taken as $\sin \omega t$, where ω is the imaginary part of the conjugate roots, the contribution to the overall response is a sinusoid with an amplitude that increases as a polynomial in time.

For eqn.⑤ for change in U same conditions as above apply. Therefore, the definition of stability for linear systems may be translated to the following criterion: "A linear control system is unstable if any roots of its characteristic equation are on, or to the right of the imaginary axis. Otherwise the system is stable.

The characteristic eqn. of a control system, which determines its stability, is the same for set-point or load changes. It depends only on $G(s)$, the open-loop transfer function.

The stability of a control system is determined solely by its open-loop transfer function through the roots of the characteristic equation.

Example -



In terms of fig., a control system has the transfer functions

$$G_1 = 10\frac{(0.5s+1)}{s} \quad \text{(PI controller)}$$

$$G_2 = \frac{1}{2s+1} \quad \text{(stirred tank)}$$

$H = 1$ (measuring element without lag)

Find the characteristic equation and its roots, and determine whether the system is stable.

→ The open-loop transfer function.

$$G = G_1 G_2 H = \frac{10(0.5s+1)}{s(2s+1}$$

The characteristic eqn. is therefore. $1+G=0$

$$1+\frac{10(0.5s+1)}{s(2s+1)} = 0$$

$$2s^2+s+5s+10 = 0$$

$$2s^2+6s+10 = 0$$

$$s^2+3s+5 = 0$$

Solving by the quadratic formula.

$$s = \frac{-\frac{3}{2} \pm \sqrt{9-20}}{2}$$

OR

$$s_1 = \frac{-3}{2} + j\frac{\sqrt{11}}{2}$$

$$s_2 = \frac{-3}{2} - j\frac{\sqrt{11}}{2}$$

Since the real part of s_1 and s_2 is negative $(-3/2)$, the system is stable.

ROUTH TEST FOR STABILITY

* The Routh test is a purely algebric method for determining how many roots of the characteristic equation have positive real parts.
* It can also used to determine whether the system is stable, for if there are no roots with positive real parts, the system is stable.
* R.Test is limited to systems that have polynomial characteristic equations.
* This means that R.Test cann't be used to test the stability of a control system containing a transportation lag.

Procedure for examining the roots:

1. Write the characteristic eqn. in the form.

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots\dots + a_n = 0 \quad \text{——— (1)}$$

where, a_0 – is positive (If a_0 is originally negative, both sides are multiplied by -1).

2. It is necessary that all the co-efficients.

$$a_0, a_1, a_2, \dots\dots\dots a_{n-1}, a_n$$

be positive if all the roots are to lie in the left half plane. If any coefficient is negative, the system is definitely unstable, and the R. test is not needed to answer the question of stability. (But R.T will give no. of roots in right half plane).

3. If all the co-efficients are positive, the system may be stable or unstable.

Date: 16.1.2021

Routh Array

Arrange the coefficients of eqn. ① into the first two rows of the Routh array.

Row				
1	a_0	a_2	a_4	a_6
2	a_1	a_3	a_5	a_7
3	b_1	b_2	b_3	
4	c_1	c_2	c_3	
5	d_1	d_2		
6	e_1	e_2		
7	f_1			
$n+1$	g_1			

The array has been filled in for $n=7$. For any other value of n, the array is prepared in the same manner.
In general, there are $(n+1)$ rows. For n even, the first row has one more element than the second row.

The elements in the remaining rows are found from the formulas,

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} \quad \ldots\ldots$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1} \quad \ldots\ldots$$

$\ldots\ldots\ldots\ldots$

The elements ~~for~~ in any row are always derived from the elements of the two preceding rows. During the computation of the Routh array, any row can be divided by a positive constant without changing the results of the test.

After getting the Routh array, the following theorems are applied to determine stability.

Theorems of the Routh Test:

1. The necessary and sufficient condition for all the roots of the characteristic eqⁿ to have negative real parts (stable system) is that all elements of the column of the Routh array (a_0, a_1, b_1, c_1 etc.) be positive and non-zero.

2. If some of the elements in the first column are -ve, the number of roots with a positive real part (in the right half plane) is equal to the number of sign changes in the first column

3. If one pair of roots is on the imaginary axis, ~~equidis~~ equidistant from the origin, and all other roots are in the left half plane, all the elements of the n^{th} row will vanish and none of the elements of the preceding row will vanish. The location of the pair of imaginary roots can be found by solving the equation

$$Cs^2 + D = 0 \quad \text{--- (a)}$$

where the coefficients C and D are the elements of the array in the $(n-1)^{th}$ row as read from left to right, respectively.

The elements ~~for~~ in any row are always derived from the elements of the two preceding rows. During the computation of the Routh array, any row can be divided by a positive constant without changing the results of the test.

After getting the Routh array, the following theorems are applied to determine stability.

Theorems of the Routh Test:

1. The necessary and sufficient condition for all the roots of the characteristic eqn to have negative real parts (stable system) is that all elements of the column of the Routh array (a_0, a_1, b_1, c_1 etc.) be positive and non-zero.

2. If some of the elements in the first column are -ve, the number of roots with a positive real part (in the right half plane) is equal to the number of sign changes in the first column.

3. If one pair of roots is on the imaginary axis, ~~equidist~~ equidistant from the origin, and all other roots are in the left half plane, all the elements of the n^{th} row will vanish and none of the elements of the preceding row will vanish. The location of the pair of imaginary roots can be found by solving the equation.

$$Cs^2 + D = 0 \quad \text{——} \quad (D)$$

where the coefficients C and D are the elements of the array in the $(n-1)^{th}$ row as read from left to right, respectively.

2. If some of the elements in the first column are negative, the number of roots with a positive real part (in the right half plane) is equal to the number of sign changes in the first column.
3. If *one* pair of roots is on the imaginary axis, equidistant from the origin, and all other roots are in the left half plane, all the elements of the nth row will vanish and none of the elements of the preceding row will vanish. The location of the pair of imaginary roots can be found by solving the equation

$$Cs^2 + D = 0 \tag{14.10}$$

where the coefficients C and D are the elements of the array in the $(n - 1)$th row as read from left to right, respectively. We shall find this last rule to be of value in the root-locus method presented in the next chapter.

The algebraic method for determining stability is limited in its usefulness in that all we can learn from it is whether a system is stable. It does not give us any idea of the degree of stability or the roots of the characteristic equation.

Example 14.2. Given the characteristic equation

$$s^4 + 3s^3 + 5s^2 + 4s + 2 = 0$$

determine the stability by the Routh criterion.

Since all the coefficients are positive, the system may be stable. To test this, form the following Routh array:

Row			
1	1	5	2
2	3	4	
3	11/3	6/3	
4	26/11	0	
5	2		

The elements in the array are found by applying the formulas presented in the rules; for example, b_1, which is the element in the first column, third row, is obtained by

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

or in terms of numerical values,

$$b_1 = \frac{(3)(5) - (1)(4)}{3} = \frac{15}{3} - \frac{4}{3} = \frac{11}{3}$$

Since there is no change in sign in the first column, there are no roots having positive real parts, and the system is stable.

In the appendix of Chap. 15, a BASIC program for computing the roots of a polynomial equation is given.

Example 14.3. (*a*) Using $\tau_1 = 1, \tau_2 = \frac{1}{2}, \tau_3 = \frac{1}{3}$, determine the values of K_c for which the control system in Fig. 14.1 is stable. (*b*) For the value of K_c for which the system is on the threshold of instability, determine the roots of the characteristic equation with the help of Theorem 3.

Solution. (*a*) The characteristic equation $1 + G(s) = 0$ becomes

$$1 + \frac{K_c}{(s+1)[(s/2)+1][(s/3)+1]} = 0$$

Rearrangement of this equation for use in the Routh test gives

$$s^3 + 6s^2 + 11s + 6(1 + K_c) = 0 \tag{14.11}$$

The Routh array is

Row		
1	1	11
2	6	$6(1+K_c)$
3	$10 - K_c$	
4	$6(1+K_c)$	

Since the proportional sensitivity of the controller (K_c) is a positive quantity, we see that the fourth entry in the first column, $6(1 + K_c)$, is positive. According to Theorem 1, all the elements of the first column must be positive for stability; hence

$$10 - K_c > 0$$
$$K_c < 10$$

It is concluded that the system will be stable only if $K_c < 10$, which agrees with Fig. 14.2.

(*b*) At $K_c = 10$, the system is on the verge of instability, and the element in the *n*th (third) row of the array is zero. According to Theorem 3, the location of the imaginary roots is obtained by solving

$$Cs^2 + D = 0$$

where C and D are the elements in the $(n-1)$th row. For this problem, with $K_c = 10$, we obtain

$$6s^2 + 66 = 0$$
$$s = \pm j\sqrt{11}$$

Therefore, two of the roots on the imaginary axis are located at $\sqrt{11}$ and $-\sqrt{11}$.

The third root can be found by expressing Eq. (14.11) in factored form:

$$(s - s_1)(s - s_2)(s - s_3) = 0 \tag{14.12}$$

where s_1, s_2, and s_3 are the roots. Introducing the two imaginary roots ($s_1 = j\sqrt{11}$ and $s_2 = -j\sqrt{11}$) into Eq. (14.12) and multiplying out the terms give

$$s^3 - s_3 s^2 + 11s - 11s_3 = 0$$

ed,

b: Submission of class Report & Mentor Report.

In reference to the above, I am herewith submitting my report and mentor report for 4 months ending

Comparing this equation with Eq. (14.11), we see that $s_3 = -6$. The roots of the characteristic equation are therefore $s_1 = j\sqrt{11}$, $s_2 = -j\sqrt{11}$, and $s_3 = -6$.

Example 14.4. Determine the stability of the system shown in Fig. 14.1 for which a PI controller is used. Use $\tau_1 = 1$, $\tau_2 = \frac{1}{2}$, $\tau_3 = \frac{1}{3}$, $K_c = 5$, and $\tau_I = 0.25$.

Solution. The characteristic equation is

$$1 + \frac{(K_c/\tau_1\tau_2\tau_3)(\tau_I s + 1)}{\tau_I s[s + (1/\tau_1)][s + (1/\tau_2)][s + (1/\tau_3)]} = 0$$

Using the parameters given above in this equation leads to

$$s^4 + 6s^3 + 11s^2 + 36s + 120 = 0$$

Notice that the order of the characteristic equation has increased from three to four as a result of adding integral action to the controller. The Routh array becomes

Row			
1	1	11	120
2	6	36	
3	5	120	
4	−108		
5	120		

Because there are two sign changes in the first column, we know from Theorem 2 of the Routh test that two roots have positive real parts. From the previous example we know that for $K_c = 5$ the system is stable with proportional control. With integral action present, however, the system is unstable for $K_c = 5$.

SUMMARY AND GUIDE FOR FURTHER STUDY

A definition of stability for a control system has been presented and discussed. This definition was translated into a simple mathematical criterion relating stability to the location of roots of the characteristic equation. Briefly, it was found that a control system is stable if all the roots of its characteristic equation lie in the left half of the complex plane. The Routh criterion, a simple algebraic test for detecting roots of a polynomial lying in the right half of the complex plane, was presented and applied to control system stability analysis. This criterion suffers from two limitations: (1) It is applicable only to systems with polynomial characteristic equations, and (2) it gives no information about the actual location of the roots and, in particular, their proximity to the imaginary axis.

This latter point is quite important, as can be seen from Fig. 14.2 and the results of Example 14.3. The Routh criterion tells us only that for $K_c < 10$ the system is stable. However, from Fig. 14.2 it is clear that the value $K_c = 9$

produces a response that is undesirable because it has a response time that is too long. In other words, the controlled variable oscillates too long before returning to steady state. It will be shown later that this happens because for $K_c = 9$ there is a pair of roots close to the imaginary axis.

In the next chapter tools will be developed for obtaining more information about the actual location of the roots of the characteristic equation. This will enable us to predict the form of the curves of Fig. 14.2 for various values of K_c. The advantage of these tools is that they are graphical and are easy to apply compared with standard algebraic solution of the characteristic equation.

There are two distinct approaches to this problem: root-locus methods and frequency-response methods. The former are discussed in Chap. 15 and the latter in Chaps. 16 and 17. These groups of chapters are written in parallel, and the reader may study one or both groups in either order. As a guide to making this decision, here are some general comments concerning the two approaches.

Root-locus methods allow rapid determination of the location of the roots of the characteristic equation as functions of parameters such as K_c of Fig. 14.1. However, they are difficult to apply to systems containing transportation lags. Also, they require a reasonably accurate knowledge of the theoretical process transfer function.

Frequency-response methods are an indirect solution to the location of the roots. They utilize the sinusoidal response of the open-loop transfer function to determine values of parameters such as K_c that keep these roots a "safe distance" from the right half plane. The actual transient response for a given value of K_c can be only crudely approximated. However, frequency-response methods are easily applied to systems containing transportation lags and may be used with only experimental knowledge of the unsteady-state process behavior.

A mastery of control theory requires knowledge of both methods because they are complementary. However, the reader may choose to study only frequency response and still be adequately prepared for most of the material in the remainder of this book. The choice of studying only root locus will be more restrictive in terms of preparation for subsequent chapters. In addition, much of the literature on process dynamics relies heavily on frequency-response methods.

PROBLEMS

14.1. Write the characteristic equation and construct the Routh array for the control system shown in Fig. P14.1. Is the system stable for (*a*) $K_c = 9.5$, (*b*) $K_c = 11$, (*c*) $K_c = 12$?

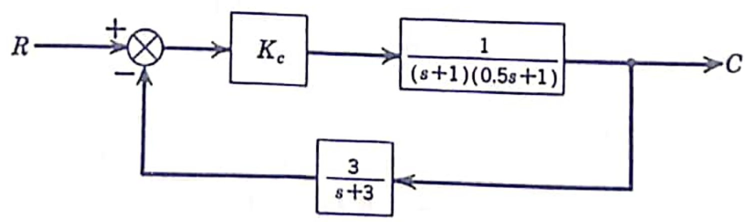


FIGURE P14-1

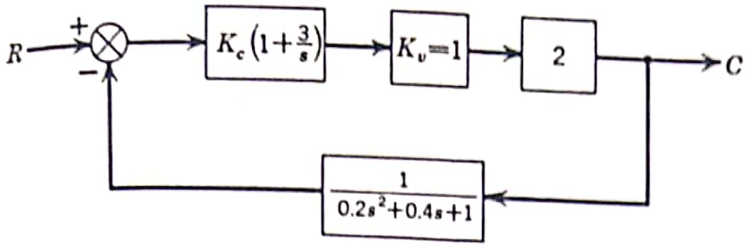


FIGURE P14-2

14.2. By means of the Routh test, determine the stability of the system shown in Fig. P14.2 when $K_c = 2$.

14.3. In the control system of Prob. 13.6, determine the value of gain (psi/° F) that just causes the system to be unstable if (a) $\tau_D = 0.25$ min, (b) $\tau_D = 0.5$ min.

14.4. Prove that, if one or more of the coefficients ($a_0, a_1, \ldots, a_n$) of the characteristic equation [Eq. (14.9)] is negative or zero, then there is necessarily an unstable root. *Hint:* First show that a_1/a_0 is minus the sum of all the roots, a_2/a_0 is plus the sum of all possible products of two roots, a_j/a_0 is $(-1)^j$ times the sum of all possible products of j roots, etc.

14.5. Prove that the converse statement of Prob. 14.4, i.e., that an unstable root implies that one or more of the coefficients will be negative or zero, is untrue for all $n > 2$. *Hint:* To prove that a statement is untrue, it is only necessary to demonstrate a single counterexample.

14.6. Deduce an extension of the Routh criterion that will detect the presence of roots with real parts greater than $-\sigma$ for any specified $\sigma > 0$.

14.7. Show that any complex number s satisfying $|s| < 1$ yields a value of

$$z = \frac{1+s}{1-s}$$

that satisfies

$$\mathrm{Re}(z) > 0$$

(*Hint:* Let $s = x + jy$; $z = u + jv$. Rationalize the fraction, and equate real and imaginary parts of z and the rationalized fraction. Now consider what happens to the circle $x^2 + y^2 = 1$. To show that the *inside* of the circle goes over to the right half plane, consider a convenient point inside the circle.)

On the basis of this transformation, deduce an extension of the Routh criterion that will determine whether the system has roots inside the unit circle. Why might this information be of interest? How can the transformation be modified to consider circles of other radii?

14.8. Given the control diagram shown in Fig. P14.8, deduce by means of the Routh criterion those values of τ_I for which the output C is stable for all inputs R and U.

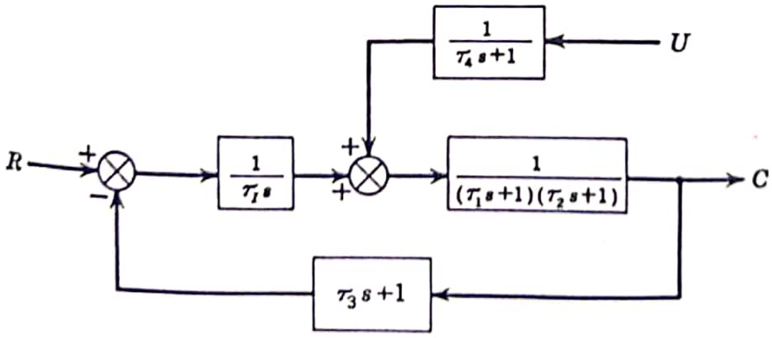


FIGURE P14-8

ROOT LOCUS

Routh's criterion was introduced to provide an algebraic method for determining the stability of a simple feedback control system from the characteristic equation of the system. This criterion also yields the number of roots of the characteristic equation that are located in the right half of the complex plane.

Root locus develops a graphical method for finding the actual values of the roots of the characteristic eqn, from which we can obtain the transient response of the system to an arbitrary forcing function.

Concept of Root Locus:

The response of the simple feedback control system a sin fig is.

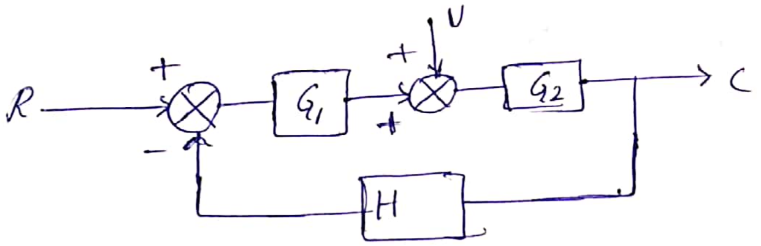


Fig - Simple feedback control system

The expression given as,

$$C = \frac{G_1 G_2}{1+G} R + \frac{G_2}{1+G} U$$

where, $G = G_1 G_2 H$

The factor in the denominator $= 1+G$

$1+G$, when set equal to zero - is called the Characteristic equation of the closed-loop system.

The roots of the characteristic equation determine the form (or character) of the response $C(t)$ to any particular forcing function $R(t)$ or $U(t)$.

The root-locus method is a graphical procedure for finding the roots of $1+G=0$, as one of the parameters of G varies continuously. The parameter that will be varied is the gain (or sensitivity) K_c of the controller.

Example – Concept of Root locus diagram

$$G_1 = K_c$$

$$G_2 = \frac{1}{(\tau_1 s+1)(\tau_2 s+1)}$$

$$H = \frac{1}{\tau_3 s+1}$$

The open-loop transfer function is

$$G = \frac{K_c}{(\tau_1 s+1)(\tau_2 s+1)(\tau_3 s+1)}$$

Alternate form.

$$G(s) = \frac{K}{(s-p_1)(s-p_2)(s-p_3)}$$

where, $K = \frac{K_c}{\tau_1 \tau_2 \tau_3}$

$$p_1 = -\frac{1}{\tau_1} \quad p_2 = -\frac{1}{\tau_2} \quad p_3 = -\frac{1}{\tau_3}$$

p_1, p_2 and p_3 are called the poles of the open-loop transfer function. A pole of $G(s)$ is any value of s for which $G(s)$ approaches infinity.

For example – If $s = p_1$, the denominator is zero and $G(s)$ approaches infinity.

Hence, $p_1 = -\frac{1}{\tau_1}$ is a pole of $G(s)$

The characteristic eqn. for the closed-loop system is,

$$1 + \frac{K}{(s-p_1)(s-p_2)(s-p_3)} = 0$$

$(s-p_1)(s-p_2)(s-p_3) + K = 0$

Using the same numerical values for the poles as $(-1, -2, -3)$ gives

$$(s+1)(s+2)(s+3) + K = 0$$

$$\therefore K = \frac{K_c}{(1)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)} = 6K_c$$

On expansion,

$s^3 + 6s^2 + 11s + (K+6) = 0$ which is third-order.

For any particular value of controller gain K_c,

if $K_c = 4.41$, $K = 26.5$

$\therefore s^3 + 6s^2 + 11s + 32.5 = 0$

Solving the eqn. for three roots gives,

$r_1 = -5.10$

$r_2 = -0.45 - j2.5$

$r_3 = -0.45 + j2.5$

Other values of K

$K = 6K_c$	r_1	r_2	r_3
0	-3	-2	-1
0.23	-3.10	-1.75	-1.15
0.39	-3.16	-1.42	-1.42
1.58	-3.45	$-1.28-j0.75$	$-1.28+j0.75$
6.6	-4.11	$-0.95-j1.5$	$-0.95+j1.5$
26.5	-5.10	$-0.45-j2.5$	$-0.45+j2.5$
60.0	-6.00	$0.0-j3.32$	$0.0+j3.32$
100.0	-6.72	$0.35-j4$	$0.35+j4$

Plot the roots r_1, r_2 and r_3 on the complex plane as K changes continuously. Such a plot is called a root-locus diagram.

There are three loci or branches corresponding to the three roots and that they "emerge" or begin (for $K=0$) at the poles of the open-loop transfer function $(-1, -2, -3)$. The direction of increasing K is indicated on the diagram by an arrow. Also the values of K are marked on each locus.

Fig-15.2
P.g. No. 180

* The root-locus diagram for this system and others to follow is symmetrical with respect to the real axis, and only the portion of the diagram in the upper half plane drawn. The characteristic equation for a physical system contains coefficients that are real, and therefore complex roots of such an equation must appear in conjugate pairs.
* The root-locus diagram has the distinct advantages of giving at a glance the character of the response as the gain of the controller is continuously changed.
* The diagram shows two critical values of K; one is at K_2 where two of the roots become equal, and the other is at K_3 where two of the roots are pure imaginary.

* The nature of the response $C(t)$ will depend only on the roots r_1, r_2, r_3. Thus, if the roots are all real, which occurs for $K < K_2$ in fig, the response will be non-oscillatory.

* If two of the roots are complex and have negative real parts ($K_2 < K < K_3$), the response will include damped sinusoidal terms, which will produce an oscillatory response.

* If $K > K_3$, two of the roots are complex and have +ve real parts, and the response is a growing sinusoid.

Example 2: Draw a root-locus diagram, let the PI controller $G_1 = K_c\left(1 + \frac{1}{T_I s}\right)$. For this case, the open-loop transfer function is.

$$G(s) = \frac{K_c (T_I s + 1)}{T_I s (T_1 s + 1)(T_2 s + 1)(T_3 s + 1)}$$

Alternate form,

$$G(s) = \frac{K(s - z_1)}{s(s - p_1)(s - p_2)(s - p_3)}$$

where, $K = \frac{K_c}{T_1 T_2 T_3}$, $z_1 = -\frac{1}{T_I}$

$p_1 = -\frac{1}{T_1}$, $p_2 = -\frac{1}{T_2}$, $p_3 = -\frac{1}{T_3}$

The term z_1 is called a zero of the open-loop transfer function. A zero of $G(s)$ is any value of s for which $G(s)$ approaches zero.

The ~~core~~ characteristic equation is,

$$1 + \frac{K(s - z_1)}{s(s - p_1)(s - p_2)(s - p_3)} = 0$$

$$s(s - p_1)(s - p_2)(s - p_3) + K(s - z_1) = 0$$

The root-locus diagram corresponds, let $T_1 = 1$, $T_2 = \frac{1}{2}$, $T_3 = \frac{1}{3}$ and $T_4 = \frac{1}{4}$. The root-locus diagram is shown in fig.

For this case there are four loci corresponding to the four roots and that they emerge (at $K=0$) from the open-loop poles $(0, -1, -2, -3)$. One of the loci moves toward the open-loop zero at -4 as K approaches infinity.

The value of $K = 3.84$, above which the roots move into the right half plane, is lower than the corresponding value of $K = 60$ for proportional control.

Fig - 15 - 3
P.g. No. 181

PLOTTING THE ROOT-LOCUS DIAGRAM

(For characteristic eqns. of any order)

* To determine the roots of the characteristic eqns. of the closed-loop control system is to write the open loop transfer function ($G = G_1 G_2 H$) in standard form.

$$G = K \frac{N}{D}$$

where, K = constant

$$N = (s - z_1)(s - z_2) \ldots\ldots\ldots (s - z_m)$$

$$D = (s - p_1)(s - p_2) \ldots\ldots\ldots (s - p_n)$$

z_i — is called zero of the open loop T.F.

p_i — is called pole of the open loop T.F.

A zero of $G(s)$ is any value of s for which $G(s)$ equals zero. The factored terms $(s - z_i)$ and $(s - p_i)$ in N/D arise naturally in the open-loop transfer function.

For example –

$$K = \frac{K_c}{T_1 T_2 T_3}$$

$$D = (s - p_1)(s - p_2)(s - p_3)$$

$$N = 1$$

Alternate form of characteristic eqn.

$$1 + G = 0$$

$$1 + \frac{KN}{D} = 0$$

$$D + KN = 0$$

Graphical method for determining the root locus

$$1 + \frac{KN}{D} = 0$$

$$\frac{KN}{D} = -1$$

In terms of the poles and zeros of the open-loop transfer function.

$$K \frac{(s-z_1)(s-z_2)\ldots\ldots\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots\ldots\ldots(s-p_n)} = -1$$

This eqn. can be written in the equivalent form involving magnitude and phase angle.

$$K \frac{|s-z_1|\,|s-z_2|\ldots\ldots|s-z_m|}{|s-p_1|\,|s-p_2|\ldots\ldots|s-p_n|} = 1 \quad \text{——— (A)}$$

$$\measuredangle(s-z_1) + \measuredangle(s-z_2) + \ldots\ldots + \measuredangle(s-z_m)$$
$$-[\measuredangle(s-p_1) + \ldots\ldots + \measuredangle(s-p_n)] = (2i+1)\pi \quad \text{——— (B)}$$

where, 'i' is any integer (Positive or Negative) or zero.

Eqns. (A) and (B) are used to find the root locus by trial and error. The trace of the locus is found entirely from the angle criterion of above eqn. (B). which is independent of K.

The gain K for any point on it may be obtained from eqn. (A) and this is magnitude criterion.

To understand the procedure for determining the root locus from the angle criterion, Consider simple example,

$$K\frac{N}{D} = \frac{K(s-z_1)}{(s-P_1)(s-P_2)}$$

for which the poles and zeros are located in fig.

Open-loop poles are shown by 'X' and open-loop zeros by 'O' in root-locus diagrams.

Let S_c — be trial point selected to plot root locus.

(S_c-z_1), (S_c-P_1) and (S_c-P_2) are vectors drawn.

If the trial point is correct, all the angles associated with these vectors (θ_1, θ_2 and α_1) substituted into eqn (B) will yield an add multiple of π.

For example,

Let the trial point S_c correct here, then

$$\alpha_1 - \theta_1 - \theta_2 = (2i+1)\pi \quad \text{for same value } i.$$

Fig. 15.4
P.g. No. 184

The trial point is moved until the angle criterion is satisfied. After a sufficient number of trial points have been established as correct, the root locus is drawn by connecting them with a smooth curve. The gains K associated with various points on the locus are determined by use of the magnitude. criterion.

For this examples,

$$\frac{K\,|S_c - z_1|}{|S_c - P_1|\,|S_c - P_2|} = 1$$

Solving for K gives,

$$K = \frac{|S_c - P_1|\,|S_c - P_2|}{|S_c - z_1|}$$

Root-locus plot is symmetrical with respect to the real axis (i.e. complex roots occur as conjugate pairs).

Scanned with CamScanner

The trial and error procedure for finding points on the loci need be done for only the upper half plane. The loci in the lower half plane can be drawn from symmetry.

Rules for Plotting Root-Locus Diagrams (Negative Feedback)

Here, $n \geq m$

Rule 1: The number of loci or branches is equal to the no. of open-loop poles, n.

Rule 2: The root loci begin at open-loop poles and terminate at open-loop zeros. The termination of $(n-m)$ of the loci will occur at the zeros at the zeros at infinity along asymptotes. In the case of a q^{th}-order pole, q loci emerge from it. For a q^{th}-order zero, q loci terminate there.

Rule 3: Locus on real axis

The real axis is part of the root locus when the sum of the number of poles and zeros to the right of a point on the real axis is odd. It is necessary to consider only the real poles and zeros in applying this rule, for the complex poles and zeros always occur in conjugate pairs and their effects cancel in checking the angle criterion for points on the real axis. Furthermore, a q^{th} order pole (or zero) must be counted q times in applying the rule.

Rule 4: Asymptotes

There are $(n-m)$ loci that approach (as $K \to \infty$) asymptotically $(n-m)$ straight lines, radiating from the center of gravity of the poles and zeros of the open-loop transfer function. The center of gravity is given by

$$\gamma = \frac{\sum_{j=1}^{n} P_j - \sum_{i=1}^{m} z_i}{n-m}$$

These asymptotic lines make angle of $\pi[(2k+1)/(n-m)]$ with the real axis and are, therefore, equally spaced at angles $2\pi/(n-m)$ to each other $(k = 0, 1, 2, \ldots\ldots n-m-1)$.

Rule 5. Breakaway Point

The point at which two root loci, emerging from adjacent poles (or moving toward adjacent zeros) on the real axis, intersect and then leave (or enter) the real axis is determined by the solution of the equation.

$$\sum_{i=1}^{m} \frac{1}{s-z_i} = \sum_{j=1}^{n} \frac{1}{s-p_j} \quad (c)$$

These loci leave (or enter) the real axis at angles of $\pm\frac{\pi}{2}$. This eqn. is solved by trial by checking it for various test points, $s = s_c$, on the real axis b/w the poles (or zeros) of interest.

For real poles or zeros, the terms in the r.s. of above eqn. are obtained by simply measuring distances along the real axis b/w the test pt. and the poles & zeros. If a pair of complex poles, $P_i = a_i \pm jb_i$ are present, add to the R.S. of eqn. (c) the term

$$\frac{2(s-a_i)}{(s-a_i)^2+b_i^2}$$

(this term accounts for both poles of the complex pair) this term is merely the result of simplifying the sum.

$$\frac{1}{s-a_i-jb_i} + \frac{1}{s-a_i+jb_i}$$

For a pair of complex zeros, add a similar term to the L.S. of eqn. (c).

Rule 6: Angle of Departure or Approach:

There are q loci emerging from each q^{th} order open-loop pole at angles determined by.

$$\theta = \frac{1}{q}\left[(2k+1)\pi + \sum_{i=1}^{m} \measuredangle(P_a - z_i) - \sum_{\substack{j=1 \\ j\neq a}}^{n} \measuredangle(P_a - P_j)\right]$$

$k = 0, 1, 2, \ldots, q-1.$

where, $P_a \rightarrow$ particular pole of order q.

Each of the m loci that do not approach the asymptotes will terminate at one of the 'm' zeros.

Scanned with CamScanner

(This term accounts for both poles of the complex pair.) This term is merely the result of simplifying the sum

$$\frac{1}{s - a_i - jb_i} + \frac{1}{s - a_i + jb_i}$$

For a pair of complex zeros, add a similar term to the left side of Eq. (15.16).

RULE 6. ANGLE OF DEPARTURE OR APPROACH. There are q loci emerging from each qth-order open-loop pole at angles determined by

$$\theta = \frac{1}{q}\left[(2k+1)\pi + \sum_{i=1}^{m} \measuredangle(p_a - z_i) - \sum_{\substack{j=1 \\ j \neq a}}^{n} \measuredangle(p_a - p_j)\right] \tag{15.17}$$

$$k = 0, 1, 2, \ldots, q - 1$$

where p_a is a particular pole of order q. Each of the m loci that do not approach the asymptotes will terminate at one of the m zeros. They will approach their particular zeros at angles

$$\theta = \frac{1}{v}\left[(2k+1)\pi + \sum_{j=1}^{n} \measuredangle(z_b - p_j) - \sum_{\substack{i=1 \\ i \neq b}}^{m} \measuredangle(z_b - z_i)\right] \tag{15.18}$$

$$k = 0, 1, 2, \ldots, v - 1$$

where z_b is a particular zero of order v. For simple poles (or zeros) on the real axis, the angle of departure (or approach) will be 0 or π.

An analog from potential theory is useful in plotting a root-locus diagram. It may be shown that the loci correspond to the paths taken by a positively charged particle in an electrostatic field which is established by poles (positive charges) and zeros (negative charges). In general, we may expect a locus to be repelled by a pole and attracted toward a zero.

Another general aid to plotting the loci is to be aware of the fact that for $n - m \geq 2$, the sum of the roots $(r_1 + r_2 + \cdots + r_n)$ is constant, real, and independent of K. This requires that motion of branches to the right be counterbalanced by the motion of other branches to the left.

Most of the open-loop transfer functions encountered in single-loop chemical process control systems will have all their poles on the real axis. In exceptional cases where the feedback path includes second-order measuring elements, such as a pressure transmitter, the open-loop transfer function will contain complex poles, but very often they will be located so far from the remaining dominant poles that they can be ignored.

These rules and guides will now be explained by applying them to specific examples.

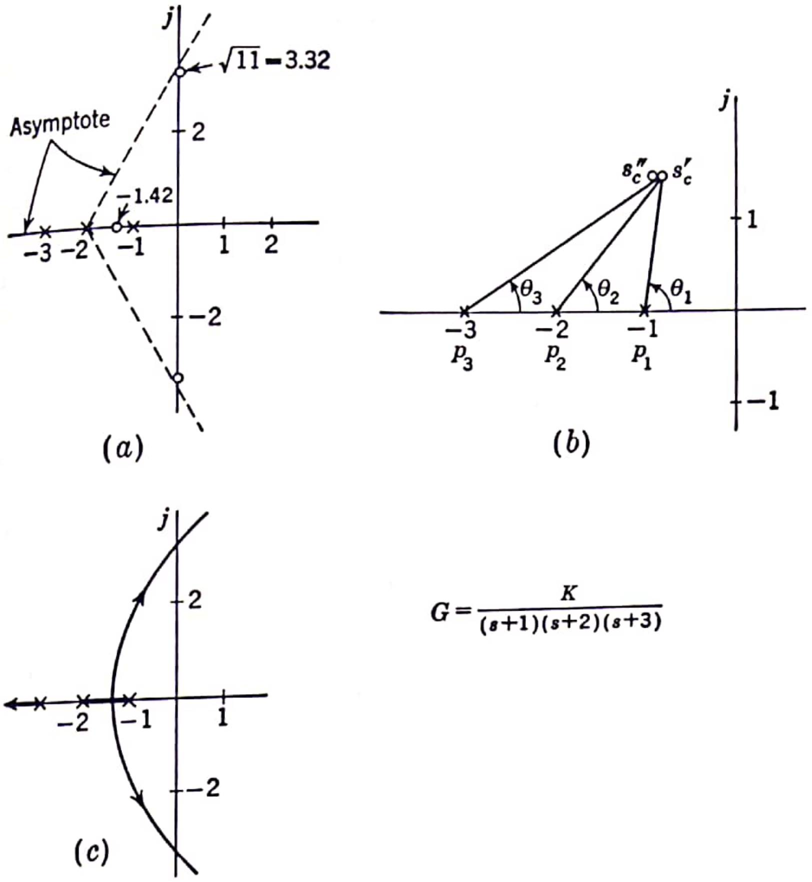


FIGURE 15-5
Root-locus construction for Example 15.1.

Example 15.1. Plot the root-locus diagram for the open-loop transfer function:*

$$G = \frac{K}{(s+1)(s+2)(s+3)}$$

In general, our stepwise procedure will follow the same order in which the rules were presented.

1. Plot the open-loop poles as shown in Fig. 15.5*a*. The poles are indicated by ×. There are no open-loop zeros for this example.
2. (Rule 1) Since we have three poles, there are three branches.
3. (Rule 3) A portion of the locus is on the real axis between -1 and -2 and another portion is to the left of -3.

*To grasp more easily the graphical procedure for plotting the root locus, the reader should actually plot these examples according to the steps given in the solution. Also note that this is the same example that was treated by algebraic methods at the beginning of this chapter.

4. (Rule 4) Since $n - m = 3$, we have three asymptotes and the center of gravity is $\gamma = (-3 - 2 - 1)/3 = -2$. Angles which the asymptotes make with the real axis are $\pi/3$, $3\pi/3$, and $5\pi/3$. These asymptotes are shown in Fig. 15.5*a*.

   With these few steps completed, a rough sketch of the root-locus diagram can be made as follows: Since the real axis to the left of -3 is an asymptote and one branch emerges from the pole at -3, it should be clear that one entire branch is the real axis to the left of the -3. Furthermore, from the fact that two loci must emerge from the poles -1 and -2 and that the real axis between these poles is part of the locus, we see that two loci move toward each other along the real axis between -1 and -2 and eventually meet at some common point. Since the location of the asymptotes is known, it is therefore necessary that the two loci that meet on the real axis must break away and eventually follow the asymptotes. From these observations, we could sketch a root-locus diagram that closely resembles that of Fig. 15.5*c*. If the breakaway point and the crossings of the imaginary axis were known, the sketch could be made with considerable accuracy. We now continue the example by applying Rule 5 to find the breakaway point and the Routh test to find the crossings of the imaginary axis.

5. *Breakaway point.* (Rule 5) The roots emerging from -1 and -2 move toward each other until they meet, at which point the loci leave the real axis at angles of $\pm\pi/2$. The breakaway point is found from Eq. (15.16) as follows

$$0 = \frac{1}{s - p_1} + \frac{1}{s - p_2} + \frac{1}{s - p_3}$$

or

$$0 = \frac{1}{s + 1} + \frac{1}{s + 2} + \frac{1}{s + 3}$$

Solving this by trial and error gives

$$s = -1.42$$

6. To find the points at which the loci cross the imaginary axis, the Routh test (theorem 3) of Chap. 14 may be used. Writing the characteristic equation $D + KN = 0$ in polynomial form gives

$$D + KN = (s + 1)(s + 2)(s + 3) + K = 0$$

or

$$s^3 + 6s^2 + 11s + K + 6 = 0$$

from which we can write the Routh array:

Row		
1	1	11
2	6	$K + 6$
3	b_1	

The theorem states that, if one pair of roots are on the imaginary axis and all others in the left half plane, all the elements of the nth row must be zero. From this we obtain for the element b_1

$$b_1 = \frac{(6)(11) - (K + 6)}{6} = 0$$

Solving for K,

$$K = 60$$

A root on the imaginary axis is expressed as simply ja. Substituting $s = ja$ and $K = 60$ into the polynomial gives

$$-ja^3 - 6a^2 + 11aj + 66 = 0$$

$$(66 - 6a^2) + (11a - a^3)j = 0$$

Equating the real part or the imaginary part to zero gives

$$a = \pm\sqrt{11} = \pm 3.32$$

Therefore the loci intersect the imaginary axis at $+j\sqrt{11}$ and $-j\sqrt{11}$.

7. Having found these general features of the root-locus plot, we can sketch the root locus. If it is desirable to have a more accurate plot of the loci, the construction is continued by the trial-and-error method described earlier in this chapter.† To illustrate the method of finding roots, suppose the trial point, $s'_c = -0.75 + 1.5j$ of Fig. 15.5*b*, is selected. This point is checked by the angle criterion [Eq. (15.14), which for this example may be written

$$\measuredangle(s + 1) + \measuredangle(s + 2) + \measuredangle(s + 3) = (2i + 1)\pi$$

or

$$\theta_1 + \theta_2 + \theta_3 = (2i + 1)\pi$$

From Fig. 15.5*b*, these angles are found to be

$$\theta_1 = 81^\circ \qquad \theta_2 = 51^\circ \qquad \theta_3 = 34^\circ$$

and we have

$$81^\circ + 51^\circ + 34^\circ = 166^\circ \neq (2i + 1)\pi$$

†Several computer software packages are now available for plotting the root-locus diagram. For example, the program CC is especially useful for root-locus plotting. Details on CC and other software packages are given in Appendix 34A (of Chap. 34). Evans (1954, 1948), who developed the root-locus method, produced an instrument for plotting root-locus diagrams called the Spirule. The Sprirule was essentially a drawing instrument that was used to add angles by rotating an arm with respect to a disk. The Sprirule, which is no longer available, is now obsolete as a result of the availability of computer programs for plotting root-locus diagrams.

Shifting the trial point horizontally to the left will increase the sum of the angles. As a second trial point, $s_c'' = -0.95 + 1.5j$ gives for the sum of the angles

$$88° + 56° + 37° = 181° \cong \pi$$

This result is sufficiently close to π, which is $(2i + 1)\pi$ with $i = 0$, and we accept the point as one on the locus. In this manner, more points on the locus can be found and a curve drawn through them.

8. *Gain.* To determine the gain at various points along the loci, the magnitude criterion [Eq. (15.13)] is used. For example, if the gain at $s = -0.95 + j1.5$ (labeled s_c'' in Fig. 15.5*b*), is wanted, we measure the distances directly with a ruler; thus

$$|s - p_1| = 1.50$$
$$|s - p_2| = 1.82$$
$$|s - p_3| = 2.52$$

(It is important to measure the vector lengths in units that are consistent with those used on the axes of the graph.)

Substituting these values into Eq. (15.13) gives

$$\frac{K}{(1.50)(1.82)(2.52)} = 1$$

or $K = (1.50)(1.82)(2.52) = 6.8$. To find the point corresponding to $K = 6.8$ on the branch along the real axis to the left of p_3 requires a trial-and-error solution if the graphical approach is used. For example, if $s = -4.5$ is tried, we obtain

$$|s - p_1| = 3.5$$
$$|s - p_2| = 2.5$$
$$|s - p_3| = 1.5$$

from which we get

$$K = (1.5)(2.5)(3.5) = 13.1$$

We see that $s = -4.5$ does not correspond to a gain of 6.8. It is therefore necessary to try other values of s greater than -4.5 until the desired value of $K = 6.8$ is obtained. Although this procedure may seem very tedious, the actual calculations go quite quickly as the reader will discover while working out this example.

We also may find the root on the real axis more directly by applying the following theorem from algebra:

The sum of the roots $(r_1 + r_2 + \cdots + r_n)$ of the nth-order polynomial equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$$

is given by

$$(r_1 + r_2 + \cdots + r_n) = -\frac{a_1}{a_0}$$

In this case, we have just found the complex roots for $K = 6.8$ to be

$$r_2, r_3 = -0.95 \pm j1.5$$

The polynomial equation is

$$(s + 1)(s + 2)(s + 3) + K = 0$$

which can be expanded into

$$s^3 + 6s^2 + 11s + (K + 6) = 0$$

According to the theorem

$$r_1 + (-0.95 + j1.5) + (-0.95 - j1/5) = -\frac{6}{1}$$

or

$$6 = -[r_1 - 2(0.95)]$$

or

$$r_1 = -4.10$$

All the detailed steps needed to plot the root locus for this problem have been discussed. The complete locus is shown in Fig. 15.5*c*. This same plot is also shown in more detail in Fig. 15.2.

Example 15.2. Consider the block diagram for the control system shown in Fig. 15.6. This system may represent a two-tank, liquid-level system having a PID controller and a first-order measuring lag. The open-loop transfer function is

$$G = K_c \frac{1 + 2s/3 + 1/3s}{(20s + 1)(10s + 1)(0.5s + 1)}$$

Rearranging this into the standard form, KN/D, gives

$$G = \frac{K(s - z_1)(s - z_2)}{s(s - p_1)(s - p_2)(s - p_3)}$$

where $K = K_c/150$

$z_1 = -0.5$

$z_2 = -1$

$p_1 = -0.05$

$p_2 = -0.1$

$p_3 = -2$

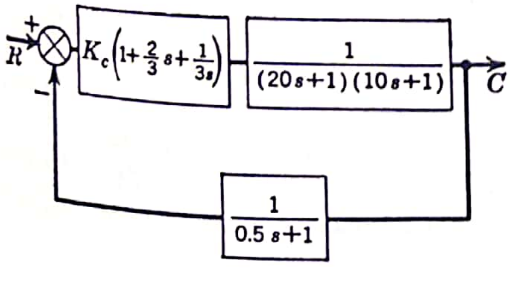


FIGURE 15-6
Block diagram for Example 15.2.

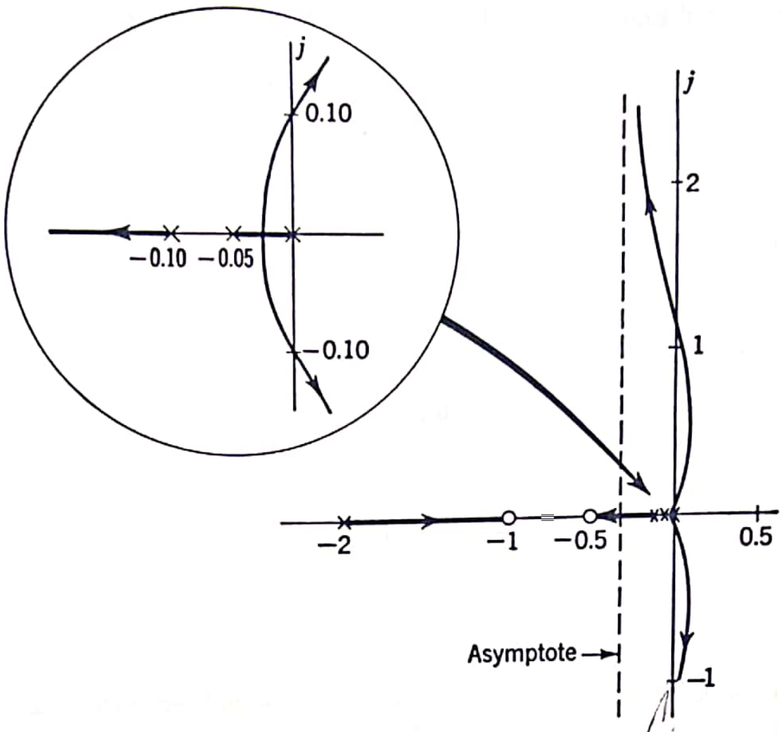


FIGURE 15-7
Root-locus diagram for Example 15.2.

In this case, there are four poles at 0, −0.05, −0.1, and −2 and two zeros at −0.5 and −1. These are plotted in Fig. 15.7. Note that the three-action controller contributes the pole at the origin and the zeros, −0.5 and −1. The steps for plotting the root-locus diagram are as follows:

1. Since there are four poles, there are four branches emerging from them.
2. Three portions of the root locus are on the real axis between 0 and −0.05, between −0.10 and −0.5, and between −1 and −2.
3. Since $n - m = 2$, there are two asymptotes, and the center of gravity is

$$\gamma = \frac{(-0.05 - 0.1 - 2) - (-0.5 - 1.0)}{2} = -0.325$$

The angles that the asymptotes make with the real axis are $\pm\pi/2$. These asymptotes are shown in Fig. 15.7.

At this stage, we can sketch part of the root-locus diagram. Since the locus is on the real axis between −0.1 and −0.5 and between −1 and −2, it should be evident that one branch moves from the pole at −2 to the zero at −1 and another branch moves from the pole at −0.1 to the zero at −0.5. The remaining two branches move from the poles at 0 and −0.05 toward each other along the real axis until they meet, at which point they must break away from the real axis and move in some way toward the vertical asymptotes that intersect the real axis at −0.325.

With the information now available, it is difficult to continue the sketch with confidence, for the breakaway point is so close to the origin that there is some likelihood that the loci will move into the right half plane before approaching the asymptote. If this should occur, each locus would have to cross the imaginary axis twice, in which case there would be an intermediate range of K over which the system is unstable. On either end of this range of K, the system is stable. This condition is called *conditional stability*. The possibility of the locus crossing the imaginary axis twice is suggested by the analog from potential theory that was mentioned earlier. This can be explained as follows: immediately after the locus leaves the real axis at the breakaway point, it has a tendency to move to the right half plane because the pole at -0.1 "repels" the locus. However, after the locus moves to a point sufficiently far from this repelling pole, it is attracted more strongly by the two zeros at -0.5 and -1 and has the tendency to return to the left half plane where we know it must eventually approach the vertical asymptote. Actually to determine whether or not the locus moves into the right half plane requires that the points at which the loci cross the imaginary axis be determined. This can be done by use of the Routh test as illustrated in Example 15.1. The details of the calculation will not be given here; however, the reader can show that there are two values of gain K which give a pair of roots of the characteristic equation that lie on the imaginary axis. These gains and corresponding roots are approximately

$$K = 0.004 \qquad \text{or} \qquad K_c = 0.6 \qquad s = \pm j0.1$$

$$K = 2.4 \qquad \text{or} \qquad K_c = 360 \qquad s = \pm j1.1$$

From these results, we conclude that the system will oscillate with constant amplitude with a frequency $\omega = 0.1$ rad/time when $K_c = 0.6$; it will also oscillate at constant amplitude with $\omega = 1.1$ when $K_c = 360$. The system is unstable for $0.6 < K_c < 360$. The system is stable for $K_c < 0.6$ and for $K_c > 360$. The complete root-locus diagram is sketched in Fig. 15.7.

SUMMARY

In this chapter, the rules for plotting root-locus diagrams have been presented and applied to several control systems. It should be emphasized that the basic advantage of this method is the speed and ease with which a rough sketch of the loci can be obtained. This sketch frequently gives much of the desired information on stability. A few further calculations of points on the locus are usually all that are necessary to obtain accurate, quantitative behavior of the roots.

The root locus for variation of parameters other than K_c, such as τ_D, has not been discussed here. The method of constructing this type of diagram is similar to that presented here and is discussed in detail in other texts [see Wilts (1960)].

Once the roots are available, the response of the system to any forcing function can be obtained by the usual procedures of partial fractions and inversion given in Chap. 3.